INTEGRABILITY OF HAMILTONIAN SYSTEM WITH HOMOGENEOUS POLYNOMIAL POTENTIAL OF DEGREE FOUR

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Abstract

The analysis of non-linear dynamical problems is important in both mathematical and physical point of view. The non-linear systems are not explicitly solvable and they are chaotic depending upon the value of the control parameters. From physical point of view, the existence of integrable nonlinear dynamical systems often means the existence of very regular motion. From mathematical point of view, they imply the existence of beautiful analytic and geometric structures. The concept of integrability is itself in a sense not well defined and seems to be no unique definition. The integrability nature of dynamical systems can be methodologically investigated using the following two broad notions (1) Integrability in the complex time plane: Painleve Property (2) Complete Integrability and Liouville Integrability. In this paper the Painleve method is used to check the integrability of Hamiltonian system. The main objective of this work is to analyses the integrability of Hamiltonian system with a homogeneous polynomial potential of degree four using Painleve test.

Key words : Painleve, Integrability, Hamiltonian

Introduction

Integrating the non-linear differential equations completely or obtaining their analytic solutions or finding integrals of motion systematically or invariants seem to be rare. What does one mean by integrability of a non-linear differential equation and when does it occur for a given system is the two fundamental questions which arise in this regard. The answer to the former question is somewhat vague as the concept of integrability of differential equation is itself in a sense not well defined and there seems to be no unique definition to it as yet. For the later, it is even more difficult to answer as no well-defined criteria seem to exist to identify integrable cases. From a qualitative point of view, integrability can be considered as a mathematical property that can be successfully used to obtain more predictive power and quantitative information to understand the dynamics of the system locally as well as globally. Investigations [1,2] which are in a sense revival of the efforts of the mathematicians and physicists of the past century show that the integrability nature of dynamical systems can be methodologically investigated using at least the following two broad notions. The first one uses essentially the literal meaning: Integrable-Integrated with sufficient number of integration constants and Non-Integrable-Proven not to be integrable. Thus loose definition of integrability can be related to the existence of single-valued, meromorphic solutions, a concept originally advocated by Fuchs [3] Kovalesvskaya [4,5] Painleve [6] and others for differential equations. Such a definition then leads to the notion of integrability in the complex time plane, which is generally called the Painleve Property. The second notion, particularly applicable to Hamiltonian systems, is to look for sufficient number of single-valued, analytic, involutive integrals of motion: N integrals for a Hamiltonian system with N-degrees of freedom, so that the associated Hamiltonians' equations of motion can be integrated by quadratures in the sense of Liouville.

Hamiltonian systems describe the movements of an object whose energy is conserved. Hamilton's equations of some Hamiltonian systems can be integrated by "quadratures", these are the completely integrable Hamiltonian systems. Example of such system is the motion of a top. The motion of a free particle on a surface of revolution or an ellipsoid are other simple examples of completely integrable Hamiltonian systems. In the special setting of Hamiltonian systems, we have the notion of integrability in the Liouville sense. For many mechanical systems, the Hamiltonian takes the form

$$H(q,p)=T(q,p)+V(q)$$

where $T(q,p)$ is the kinetic energy, and $V(q)$ is the potential energy of the system. Such systems are called the natural Hamiltonian systems.

A dynamical system is integrable when it can be solved in some way. One restrictive way in which this can happen is if the flow of the vector field can be constructed analytically. However, since this can almost never be done and this is not an especially useful class of systems. However, there is a class of Hamiltonian system action-angle systems, whose solutions can be obtained analytically, and there is a well-accepted definition of integrability for Hamiltonian dynamics due to Liouville.

Singularity Point Analysis for Ordinary Differential Equations

For an ordinary differential equation to be of P-type, it is necessary that it has no movable branch points, either algebraic or logarithmic. We do not consider in our following analysis the presence of essential singularities,whose treatment appears to be much more complicated and the theory is probably not complete to locate them. In the following, we describe the ARS [7] which provides a systematic way to investigate the presence of movable critical points of branch point type and to determine whether the given ordinary differential equation is of P-type or not.

To be specific, we consider an n^{th} order ordinary differential equation

$$\frac{d^n w}{dz^n} = F(z; w, \frac{dw}{dz}, \dots \dots \dots \frac{d^{n-1}w}{dz^{n-1}}) \text{ --------------- } (1)$$

Here F is analytic in z aand rational in their other arguments.We look for a solution of equation(1) as a Laurent series in the neighbourhood of a movable singular point $z_{0.}$Then the ARS algorithm essentially consists of the following three steps.

1. Determination of leading -order behaviours of the Laurent series in the neighbourhood of the movable singular point $z_{0.}$
2. Determination of resonances, that is, the powers at which arbitrary constants of the solution of equation (1) can enter into the Laurent series expansion.
3. Verifying that sufficient number of arbitrary constants exist without the introduction of movable critical points.

At the end of the above three steps one will be in a position to check the necessary conditions for the existence of P-type solution and integrability of equation (1).

Painleve Analysis and Integrability of Hamiltonian System with Homogeneous Polynomial Potential of Degree Four

The Hamiltonian reads

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}ax^2(x^2 - y^2 + 2xyi) + \frac{1}{4}(x^4 + y^4 + 2x^2y^2) \text{-----}$$

(2) where is a constant.

It is well-known that the above Hamiltonian system is widely used as a models in lattice dynamics, condensed matter theory, field theory, astrophysics etc.

Now the Hamiltons' equations read,

$\frac{d^2x}{d^2t} = -4x^3\alpha - 2\beta xy^2 - 3\gamma x^2y - x^3 + xy^2$----------(3)

$\frac{d^2y}{dt^2} = -2\beta x^2y - \gamma x^3 - y^3 + x^2y$------------------------(4)

where $\alpha = \frac{a}{2}$, $\beta = \frac{-a}{2}, \gamma = ai$

Leading - order Behaviour

Let us consider the equations 3 and 4, we assume that the leading order behaviour of x(t) and y(t) in a sufficiently small neighbourhood of the movable singularity t_0 is

$x(t) \approx a_0\, \tau^p$ and $y(t) \approx b_0\tau^q\tau = (t - t_{0)} \to 0$---------(5)

Substituting equation (5) in equation (3) and (4), we get

$a_0p(p-1)\tau^{p-2} + 4\alpha a_0^3\tau^{3p} + 2\beta a_0b_0^2\tau^{p+2q} + 3\gamma a_0^2b_0\tau^{2p+q} + a_0^3\tau^{3p} + a_0b_0^2\tau^{p+2q} = 0$ --- (6)

b_0q(q-1)$\tau^{q-2} + 2\beta a_0^2b_0\tau^{2p+q} + \gamma a_0^3\tau^{3p} + b_0^3\tau^{3q} + a_0^2b_0\tau^{2p+q} = 0 - - - -(7)$

From the above equations it is clear that each term is a leading-order term and so we obtain,

p = -1 and q = -1 and

Resonances

For finding the resonances, we substitute

$2a_0 + 4\alpha a_0^3 + 2\beta a_0b_0^2 + 3\gamma a_0^2b_0 + a_0^3 + a_0b_0^2 = 0$---------(8)

$2b_0 + 2\beta a_0^2b_0 + \gamma a_0^3 + b_0^3 + a_0^2b_0 = 0$------------------------(9)

Into terms of equations (3) and (4)

Retaining only the leading-order terms, we obtain a system of 2-coupled linear algebraic equations, in matrix form it can be written as

$M_2(r)\Omega = 0, \Omega = (\Omega_1, \Omega_2)^T$

where is a 2 matrix is given by

$$\begin{pmatrix} r^2 - 3r + 8\alpha a_0^2 + 3\gamma a_0 b_0 + 2a_0^2 & 4\beta a_0 b_0 + 3\gamma a_0^2 + 2a_0 b_0 \\ 4\beta a_0 b_0 + 3\gamma a_0^2 + 2a_0 b_0 & r^2 - 3r + 2 + 2\beta a_0^2 + 3b_0^2 + a_0^2 \end{pmatrix}$$

For non-trivial set of solutions (we demand

(Ω_1, Ω_2) we demand *det* M_2 (r)^2

$(r^3 - 3r + 8\alpha a_0^2 + 3\gamma a_0 b_0 + 2a_0^2)(r^2 - 3r + 2 + 2\beta a_0^2 + 3b_0^2 + a_0^2) - (4\beta a_0 b_0 + 3\gamma a_0^2 +$

$2a_0 b_0)^2 = 0.$

The ARS algorithm demands the above quartic polynomial admits a root equal to -1. This is possible only if α = 0, β = 0, γ = 0 which implies α = 0

Case 1: a = 0

The associated equations of motion are

$$\frac{d^2x}{d^2t} + x^3 + xy^2 = 0 \text{---------} 11$$

$$\frac{d^2y}{dt^2} + y^3 + x^2y = 0 \text{---------} 12$$

Leading-Order Behaviour

Consider the equations (11) and (12), we assume that the leading-order behaviour of *x*(*t*) and *y*(*t*) b_0 τ^q in a sufficiently small neighbourhood of the movable singularity t_0 is

$x(t) \approx a_0 \tau^p$ and $y(t) \approx b_0 \tau^q, \tau = (t - t_{0)} \rightarrow 0$--------13

To determine p, q, we use equation (13) in equation (11) and (12) and obtain a pair of leading - order equations,

$a_0 p(p-1)\tau^{p-2} + a_0^3 \tau^{3p} + a_0 b_0^2 \tau^{p+2q} = 0$---------------------14

$b_0 q(q\text{-}1)\tau^{q-2} + b_0^3 \tau^{3q} + a_0^2 b_0 \tau^{2p+q} = 0$-------------------------15

Equating the powers we get

p = -1 and q = -1, substitute this values we get,

$$a_0^2 + b_0^2 = -2$$

So we cannot take as arbitrary. Here all the terms are leading-order terms.

Resonances

For finding the resonances, we substitute into equations (11) and (12)

$x(t) \approx a_0 \tau^p + \Omega_1 \tau^{p+r}$ and $y(t) \approx b_0 \tau^q + \Omega_2 \tau^{q+r}$-----(16)

Retaining only the leading-order terms,we obtain a system of linear algebraic equations

$M_2(r)\Omega = 0,\ \Omega = (\Omega_1, \Omega_2)^T$

whereM_2 (r) is a 2 × 2 matrix dependent on r, the form of $M_2(r)$ is,

$$\begin{pmatrix} r^2 - 3r + 2a_0^2 & 2a_0 b_0 \\ 2a_0 b_0 & r^3 - 3r + 2b_0^2 \end{pmatrix}$$

For non-trivial set of solutions (Ω_1, Ω_2 we demand *det* $M_2(r)$ is, On solving we will get r = -1,0,3,4

Excluding -1 and possibly 0, all the remaining roots are positive real integers,then there are no algebraic branch points.So the given ordinary differential equation is of P-type.

Conclusion

From this analysis we can conclude that Hamiltonian System with Homogeneous Polynomial Potential of Degree Four is of P-type with the parametric restriction a=0, which leads to the notion of integrability in the complex plane.

References

[1] D. F Escande , *Stochasticity in Classical Hamiltonian Systems:* Universal Aspects,-Phy.Rep(1985),121,165.

[2] M.Lakshmanan, *Solitons: Introduction and Applications, Springer-Berlin(1988)*

[3] L.Fuchs, *Uber Differentialgleichulgen Dereniltegrate feste Verzweigungspunkte Besitzeh,Sitz.Akad.Wiss Berlin*,32,(1884),699.

[4] S.Kovalevskaya, *Acta Math(stockh),12(1889),177.*

[5] S.Kovalevskaya, *Acta Math(stockh),14(1889),81.*

[6] Painleve,*Bull* de *Soc,Math 28,(1900),201.*

[7] M.J Ablowitz,A.Ramani and H Segur Lett.al Nuovo Cimento,23(1978)333,J.Math-Phys 21(1980)715,21(1980)1006.