A STUDY ON STRUCTURE GRACEFUL INDEX

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ABSTRACT

In this presentation I introduce an upper bound for structure graceful index of Kn.

Today, graph theory is one of the most flourishing branches of Mathematics and has applications to a wide variety of subjects. .Also, graceful labelling plays a vital role. But the complete graph Knis not graceful for $n>4$. This paper is a brief study on graceful labelling of graph and structure graceful index of graph. This paper investigates about the graceful labelling of graphs and trees. It is clearly demonstrated that all wheels are graceful for all $n \geq 3$. Also it is illustrated that Petersen graph, hypercubes and caterpillars are all graceful.AGn graph which is graceful for $n>4$ has $V(G_n)=\{v_1,v_2,\ldots\ldots,v_n\}$ and $E(G)=\{v_1 v_i / i>1\} U \{v_2 v_i / i>3\} U \{v_j v_n / 5 \leq i \leq n\}$,for $n>4$.

Keywords: Graceful labelling, Graph structure, Structure graceful index, k-Structure graceful.

1. INTRODUCTION

In many real life situations, we are using complete graphs. Also, graceful labelling plays a vital role. Let G be an undirected graph without loops or double connections between vertices. In labelling (valuation or numbering) of a graph G, we associate distinct nonnegative integers to the vertices of G as vertex labels (vertex values or vertex numbers) in such a way that each edge receives a distinct positive integer as an edge label (edge value or edge number) depending on the vertex labels of vertices which are incident with this edge. Graph labelling gave birth to families of graphs with attractive names such as graceful, felicitous and elegant. The name "graceful labelling" is due to Solomon W. Golomb; this class of labelling was originally given the name β-labelling by Alexander Rosa in a 1967 paper on graph labelling. But the complete graph K n is not graceful for n > 4. In the course of the proof, we found a graph Gn , which is graceful for n>4. A Gn graph has V(G n) ={v 1 ,v 2 ,…,v n } and E(Gn)={v 1 v i / i>1} U {v2 v i /i>2} U {v3 v i / i>3} U {vjvn/5≤i<n}, for n>4. Using this G n graph, we find the upper bound for the structure graceful index of Kn ,n>10.

Although numerous families of graceful graphs are known, a general necessary or sufficient condition for gracefulness has not yet been found. Also it is not known if all tree graphs are graceful. Another important labelling is an α -labelling or α -valuation which was also introduced by Rosa . An α-valuation of a graph G is a graceful valuation of G which also satisfies the following condition: there exists a number γ (0 ≤ γ < E(G)) such that, for any edge e∈ E(G) with the end vertices u, v ∈ V(G), min { vertex label (v), vertex label (u) } ≤ γ < max { vertex label (v), vertex label (u) }

It is clear that if there exists an α-valuation of graph G, then G is a bipartite graph. During the past thirty years, over 200 papers on this topics have been appeared in journals. Although the conjecture that all trees are graceful has been the focus of many of these papers, this conjecture is still unproved. Unfortunately there are few general results in graph labelling. Indeed even for problems as narrowly focused as the ones involving the special classes of graphs, the labelling have been hard-won and involve a large number of cases.

2. STRUCTURE GRACEFUL LABELLING

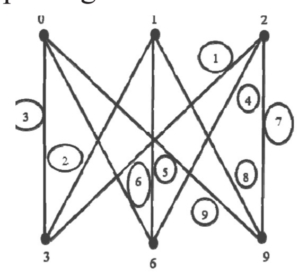
2.1GRACEFUL LABELLINGS

A graceful labelling of a graph G with m edges is a function f: V(G)→{0, 1, 2,……..m} such that distinct vertices receive distinctnumbers and {|f (u) – f(v) | : uv ε E(G)}={1, 2, ……m}.

A graph is graceful if it has a graceful labelling. In order for a graphto be graceful, it must be without loops or multiple edges.

Examples 1

Thomsen Graph



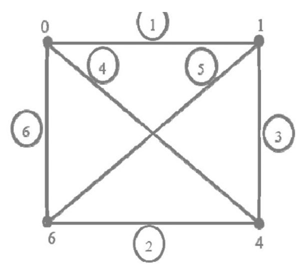


Figure 1 In all the figures, vertices are labeled as numbers and graceful labels of edges are marked inside small circles.

2.2 STRUCTURE GRACEFUL INDEX

The structure graceful index of a graph G is defined as the minimum k for which G is k-structure graceful. Let us denote it by SGI(G).

2.3 k-STRUCTURE GRACEFUL

A graph G = (V,E) is said to be k-structure graceful if E can be partitioned into k disjoint subsets E 1 ,E 2 ,…,E k such that the graph structure (V(G), E 1 , E 2 , …, E k) is graceful.

THEOREM 2.1:

The wheel Wn is graceful for all n≥3.

Proof:

Construct a numbering f of a wheel Wn = Cn +K1 , as follows: f assigns the vertex of K1, the number 0 and f assigns the vertices of the cycle Cn consecutively the numbers

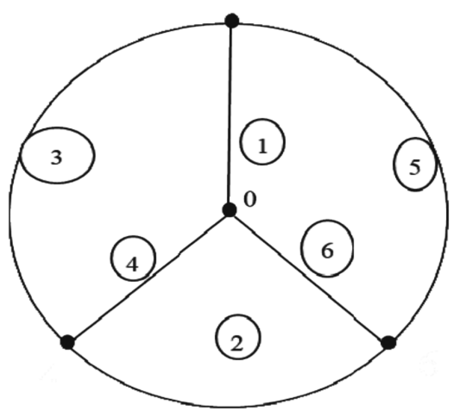
(i) 2_n, 1, $_{2n-3}$, 3, 2_{n-5}, 5 …n+2, n-2, 2_n-2, 2_n if n is a odd.

(ii) 2_n, 1, 2_{n-3}, 3, 2_{n-5}, 5,……n-3, n+1, 2, 2n if n is even.

In order to show that numbering f is graceful, it suffices to verify that for every number i ε {1, 2,………2_n} there is exactly one edge numbered i. All the verifications are trivial.

Case 1: When n is odd

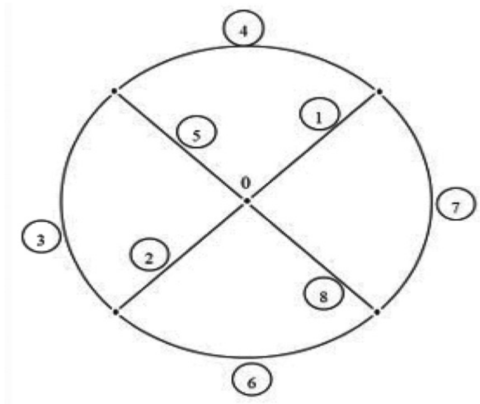
Figure 2



iε {1, 2, 3, 4, 5, 6}

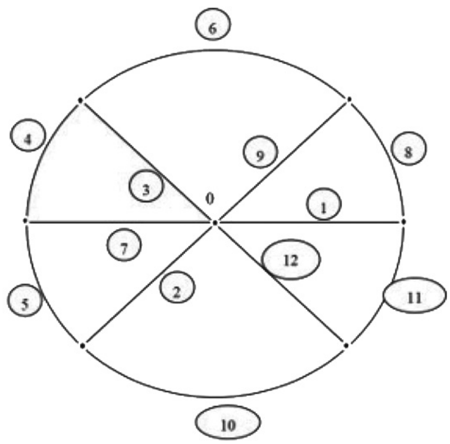
W_3 (n=3)

Case 2: when n is even



{1, 2, 3, 4, 5, 6, 7, 8,}

W_6 (n=6)



iεiε {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

W_4 (n = 4)

Figure 3

MORE EXAMPLES OF GRACEFUL GRAPHS

- PETERSEN GRAPH
- HYPERCUBES
- CATERPILLARS

CATERPILLARS ARE GRACEFUL

A caterpillar is a tree having a path that contains at least one vertex of every edge. (It can be taken to be a path of maximum length) Every caterpillar has a graceful labeling. The illustration below shows a caterpillar with a graceful labeling.

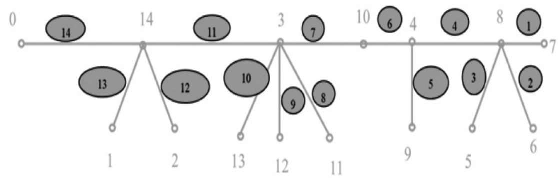


Figure 4

THEOREM 2.2:

$$SGI(K_n) \le \begin{cases} \left[\frac{n-5}{4}\right]+1, & \text{when } n \equiv 1 \pmod 4 \\ \left[\frac{n-6}{4}\right]+1, & \text{when } n \equiv 2 \pmod 4 \\ \left[\frac{n-7}{4}\right]+2, & \text{when } n \equiv 3 \pmod 4 \\ \left[\frac{n-8}{4}\right]+2, & \text{when } n \equiv 0 \pmod 4 \end{cases}$$

where $n > 10$.

To prove this theorem we need the following lemma.

Lemma: One point union of Km and K1,n is graceful for

$2 < m < 7$.

Case (i): When m = 3

Define

$$f(v_i) = \begin{cases} 0, & i = 1 \\ i, & i > 1 \end{cases}$$

Case (ii): When m = 4

$$\text{Define } f(V_i) = \begin{cases} 0, & i=1 \\ 6, & i=2 \\ 5, & i=3 \\ 2, & i = 4 \\ i + 2, & 5 \le i \le 4+n \end{cases}$$

Case (iii): When m = 5

$$\text{Define } f(v_i) = \begin{cases} 0, & i = 1 \\ 11, & i = 2 \\ 10, & i = 3 \\ 2, & i = 4 \\ 7, & i = 5 \\ 6, & i = 6 \\ i + 5, & 7 \le i \le 5+n \end{cases}$$

case (iv): When m = 6

$$\text{Define } f(v_i) = \begin{cases} 0, & i = 1 \\ 17, & i = 2 \\ 16, & i = 3 \\ 2, & i = 4 \\ 13, & i = 5 \\ 7, & i = 6 \\ 8, & i = 7 \\ 12, & i = 8 \\ i + 9, & 9 \le i \le 6+n \end{cases}$$

Proof for the theorem: Partition the edges of K_n ie) $E(K_n)$ into two sets namely, $E(G_n)$ and $E(K_n \setminus G_n)$, then $E(K_n \setminus G_n)$ into $E(G_n{-}4)$ and $E(\ \setminus G_n{-}4)$, then $E(\ \setminus G_n{-}4)$ into $E(G_n{-}8)$ and $(E(\ \setminus G_n{-}8)$, then $(E(\ \setminus G_n{-}8)$ into $E(G_n{-}12)$ and and so on.

From this partition, in the last step we arrive the following cases:

(i) $E(G_9)$ and edges in one point union of K_5 and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor -1}$, when $n \equiv 1 \pmod 4$

(ii) $E(G_{10})$ and edges in one point union of K_6 and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor -1}$, when $n \equiv 2 \pmod 4$

(iii) $E(G_7)$ and edges in one point union of K_3 and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor -1}$, when $n \equiv 3 \pmod 4$

(iv) $E(G_8)$ and edges in one point union of K_4 and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor -1}$, when $n \equiv 0 \pmod 4$

When $n \equiv 1 \pmod 4$:

We have subgraphs which contain the edges of $G_n, G_{n-4}, G_{n-8}, \ldots, G_9$ and one point union of K_5 and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor -1}$. Totally we have m + 1 = +1 graceful subgraphs.

$\therefore$ $SGI(K_n) \leq \ + 1$.

When $n \equiv 2 \pmod 4$:

We have sub graphs which contain the edges of $G_n, G_{n-4}, G_{n-8}, \ldots, G_{10}$ and one point union of K_6 and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor -1}$.

We have m + 1 = +1 graceful subgraphs.

$\therefore$ $SGI(K_n) \leq \left\lceil \frac{n-5}{4} \right\rceil + 1$.

When $n \equiv 2 \pmod 4$:

We have sub graphs which contain the edges of $G_n, G_{n-4}, G_{n-8}, \ldots, G_{10}$ and one point union of K_6 and $K_{1,\left\lfloor \frac{n}{4} \right\rfloor -1}$.

We have m + 1 = +1 graceful subgraphs.

$\therefore$ $SGI(K_n) \leq \left\lceil \frac{n-6}{4} \right\rceil + 1$.

When n 3(mod 4):

We have sub graphs which contain the edges of G_n, G_{n-4}, G_{n-8}, …, G_7 and one point union of K_3 and $K_{1,\left\lceil \frac{n}{4} \right\rceil -1}$.

We have m + 2 = $\left[\frac{n-6}{4}\right]$ +2 graceful subgraphs.

$\therefore$ SGI(K_n) $\leq \left[\frac{n-7}{4}\right] + 2$

When n ≡ 0 (mod 4):

In this form we have subgraphs which contain the edges of G_n, G_{n-4}, G_{n-8}, …, G_8 and one point union of K_4 and .We have m + 2 = $\left[\frac{n-6}{4}\right]$ +2 graceful subgraphs.

$\therefore$ SGI(K_n) $\leq \left[\frac{n-8}{4}\right]$ + 2.

Hence the proof.

Illustration: The 3–structure graceful labeling of K_{13} is shown below:

Here we have 13 ≡ 1(mod 4). Hence by the above result, SGI (K_{13}) = $\left[\frac{n-5}{4}\right]$ + 1 = 2 + 1 = 3. For, partition the edges of K_{13} into E(G13) and E($K_{13} \setminus G_{13}$). We have

G_{13} $\qquad$ Kn $\setminus G_n = K_{13} \setminus G_{13}$

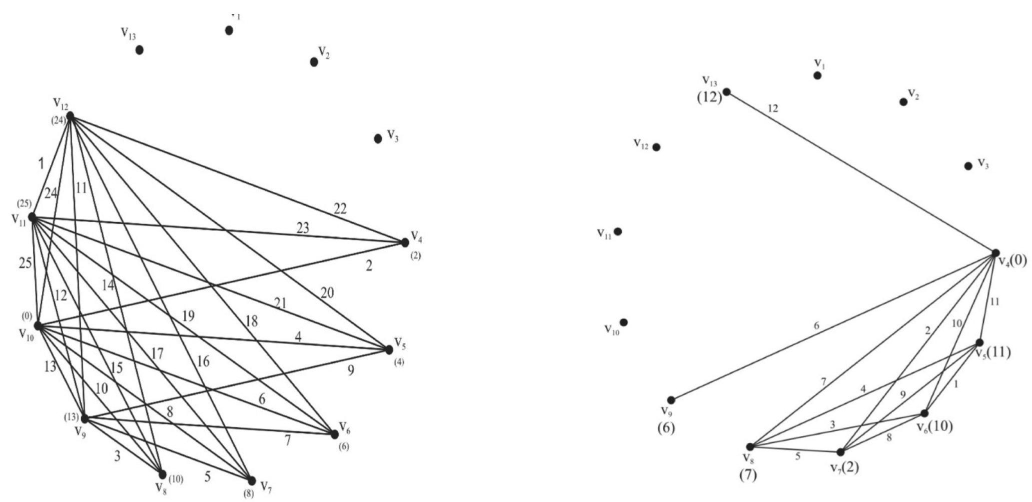


Figure 5

Partition the edges of K13 \ G13 into E(Gn–4 = G9) and E(\ G9)

G9 \ G9 = One point union of K5 and K1,2

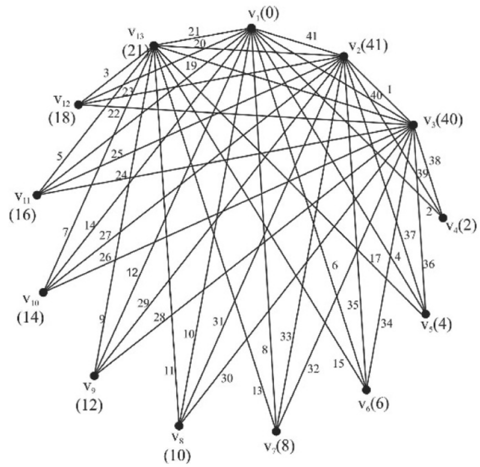


Figure 6

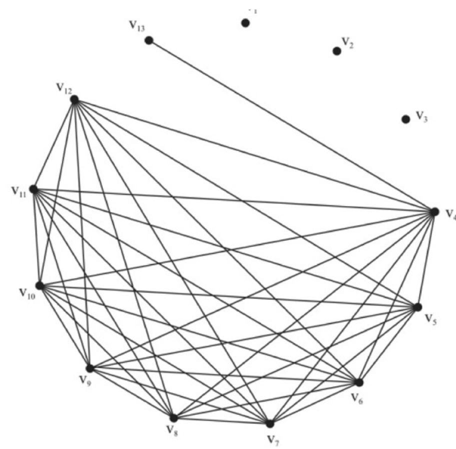


Figure 7

Hence SGI (K_{13}) = 3.

3–structure graceful labelling of K13

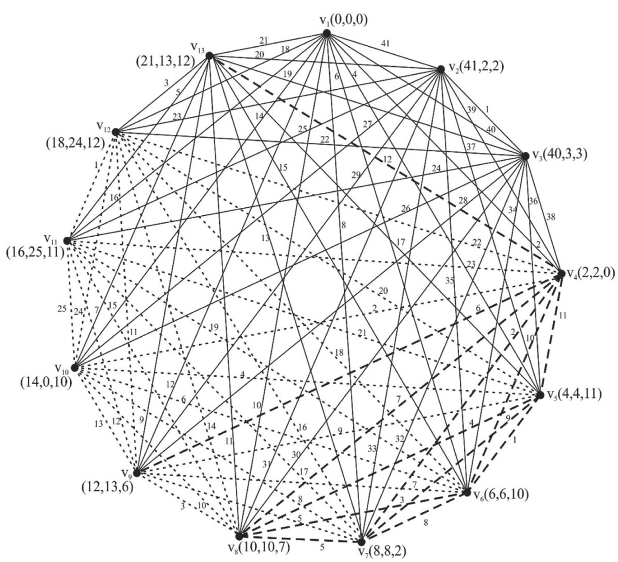


Fig.8

3. Acknowledgement

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4.Conclusion

Graceful labelling is a useful branch of labelling as it has manyimportant applications in the modern world like Optical MPLS (MultiProtocol Label Switching) networks, coding in computerprogrammes and so on. Graceful graphs are related to manymathematical topics including Golomb rulers, permutations and other graph labelling problems. They have practical applications such as radioastronomy,crystallographyetc. So we conclude with the note that the study on graceful labelling remains relevant for the times. Decomposition of complete graph Kn into graceful subgraphs has been got for n > 10. This work may contribute much on application side. The sharpness of upper bounds for SGI(Kn) is yet to be tested. The extension of this sort of work to other important families of graphs such as Petersen graphs, etc. is our next target.

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